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STEADY MOTIONS AND RELATIVE EQUILIBRIA OF MECHANICAL SYSTEMS WITH SYMMETRY†

A. V. KARAPETYAN and S. Ya. STEPANOV

Moscow

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Conservative and dissipative systems admitting continuous symmetry groups are considered. The correspondence of invariant sets of steady motions of such systems with the appropriate invariant sets of relative equilibria is discussed, as is the relationship between the stability conditions for these invariant sets.

1. Consider a conservative mechanical system with n degrees of freedom that admits of a k-parameter group of symmetries. Based on general theorems of mechanics (see, for example, [1]), such systems admit of $k + 1$ first integrals: an energy integral

$$H=\frac{1}{2}\mathbf{v}^T\mathbf{A}(\mathbf{r})\mathbf{v}+V(\mathbf{r})=\text{const} \tag{1.1}$$

and Noether integrals

$$\mathbf{K} = \mathbf{v}^T\mathbf{B}(\mathbf{r}) = \mathbf{c} = \text{const} \tag{1.2}$$

where $\mathbf{v}$ is the n-dimensional column vector of quasi-velocities and $\mathbf{r}$ is the m-dimensional column vector of quasi-coordinates, on which H and $\mathbf{K}$ depend ($\mathbf{v} \in \mathbf{R}^n$, $\mathbf{r} \in \mathbf{M} \subset \mathbf{R}^m$, $n \geqslant \dim \mathbf{M}$; $\mathbf{M}$ is the configuration manifold of the system), $\mathbf{A}(\mathbf{r})$ is the $n \times n$ kinetic energy matrix, which is positive-definite for any $\mathbf{r} \in \mathbf{M}$, $V(\mathbf{r})$ is the potential, $\mathbf{B}(\mathbf{r})$ is the $n \times k$ coefficient matrix of the Noether integrals, $\mathbf{c}$ is the k-dimensional column vector of constants of the integrals, and the superscript T stands for transposition. By Routh's theorem [2] (see also [3–6]), the critical points of the energy integral for fixed values of the other integrals correspond to steady motions (SM) of the system. Taking the structure of the integrals (1.1) and (1.2) into account, one can reduce the problem of determining these SM to that of seeking the critical points of the effective potential [7]

$$W_\mathbf{c}(\mathbf{r})=\min_\mathbf{v} H(\mathbf{v},\mathbf{r})\Big|_{\mathbf{K}(\mathbf{v},\mathbf{r})=\mathbf{c}}=V(\mathbf{r})+C(\mathbf{r})$$
$$\left(C(\mathbf{r})=\frac{1}{2}\mathbf{c}(\mathbf{D}(\mathbf{r}))^{-1}\mathbf{c}^T,\ \ \mathbf{D}(\mathbf{r})=\mathbf{B}^T(\mathbf{r})\mathbf{A}^{-1}(\mathbf{r})\mathbf{B}(\mathbf{r})\right) \tag{1.3}$$

on the configuration manifold $\mathbf{M}$.

Thus, the SM of the system have the form

$$\mathbf{r}=\mathbf{r}_\mathbf{c}^\circ,\quad \mathbf{v}=\mathbf{v}_\mathbf{c}^\circ \tag{1.4}$$

The quantities $\mathbf{r}_\mathbf{c}^\circ$ are determined from the system

$$\delta W_\mathbf{c}|_{\mathbf{r}\in\mathbf{M}}=0 \tag{1.5}$$

and $\mathbf{v}_\mathbf{c}^\circ$ from the relations

$$\mathbf{v}_\mathbf{c}^\circ=\mathbf{A}^{-1}\mathbf{B}\mathbf{D}^{-1}|_{\mathbf{r}=\mathbf{r}_\mathbf{c}^\circ}\mathbf{c}^T \tag{1.6}$$

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By the Routh–Salvadori theorem on the stability of SM [2–6] and its inverse [8–11], the SM (1.4) are stable if the effective potential reaches a strict minimum at the point $\mathbf{r}_c^\circ$, and unstable if the determinant of the quadratic form

$$\delta^2 W_c|_{\mathbf{r}=\mathbf{r}_c^\circ} \tag{1.7}$$

is negative. Incidentally, if this determinant does not vanish, the index of the quadratic form (1.7) is known as the Poincaré instability degree of the SM (1.4).

By Routh's theorem, a steady motion is stable if the Poincaré degree of instability is zero, and by Kelvin's theorem it is unstable if the degree is odd. When the degree of instability is even, gyroscopic stabilization may occur which is non-secular since it is disturbed by dissipative forces; one then says that the steady motion is unstable in the secular sense.

2. Let us assume that the system, in addition to the potential forces that are derivatives of the function is subject to $V(r)$, is subject to control forces which ensure that for all motions of the system, not only the SM considered in Section 1, the following relations are satisfied

$$\mathbf{v}^T\mathbf{B}\mathbf{D}^{-1} = \boldsymbol{\omega} \tag{2.1}$$

where $\boldsymbol{\omega}$ is a k-dimensional row vector of constants. A motion for which condition (2.1) holds and $\mathbf{r}$ = **const** will be called a relative equilibrium (RE) of the system. Thus, the REs have the form

$$\mathbf{r} = \mathbf{r}_{\boldsymbol{\omega}}^\circ, \quad \mathbf{v} = \mathbf{v}_{\boldsymbol{\omega}}^\circ \tag{2.2}$$

The quantities $\mathbf{r}_{\boldsymbol{\omega}}^\circ$ are determined from the system

$$\delta W_{\boldsymbol{\omega}}|_{\mathbf{r}\in\mathbf{M}} = 0 \tag{2.3}$$

and $\mathbf{v}_{\boldsymbol{\omega}}^\circ$ from the relations

$$\mathbf{v}_{\boldsymbol{\omega}}^\circ = \mathbf{A}^{-1}\mathbf{B}|_{\mathbf{r}=\mathbf{r}_{\boldsymbol{\omega}}^\circ}\boldsymbol{\omega}^T \tag{2.4}$$

Here

$$W_{\boldsymbol{\omega}}(\mathbf{r}) = V(\mathbf{r}) - \Omega(\mathbf{r}), \quad \Omega(\mathbf{r}) = 1/2\boldsymbol{\omega}\mathbf{D}(\mathbf{r})\boldsymbol{\omega}^T \tag{2.5}$$

is the reduced potential of the system, which includes additional forces.

By Lagrange's theorem on the stability of equilibria and its inverse (see [8, 9, 12]), the RE (2.2) is stable if the reduced potential (2.5) reaches a strict minimum at the point $\mathbf{r}_{\boldsymbol{\omega}}^\circ$, and unstable if the determinant of the quadratic form

$$\delta^2 W_{\boldsymbol{\omega}}|_{\mathbf{r}=\mathbf{r}_{\boldsymbol{\omega}}^\circ} \tag{2.6}$$

is negative. If this determinant does not vanish, the index of the quadratic form (2.6) may be defined, in a manner analogous to the Poincaré degree, as the degree of instability of the RE (2.2).

Thus, a RE is stable if the Poincaré degree of instability is zero, and unstable if it is odd. If the degree of instability is even, the RE is unstable in the secular sense, but it may be stable in Lyapunov's sense in the case of gyroscopic stabilization.

3. The problems of determining the steady motions and relative equilibria of systems with symmetry are in a sense equivalent. The rigorous formulation is as follows.

Theorem 1. For any set $\mathbf{c}$ of constants of Noether integrals (for any set $\boldsymbol{\omega}$ of constants determining the velocity of motion of the system along the symmetry group), a set of constants $\boldsymbol{\omega}$ (a set of constants $\mathbf{c}$) exists such that the solution of system (2.3) is identical with that of system (1.5): $\mathbf{r}_{\boldsymbol{\omega}}^\circ = \mathbf{r}_c^\circ$.

Proof. Let us assume that the configuration space M is defined by the relations

$$\mathbf{F}(\mathbf{r}) = \mathbf{0} \quad (\mathbf{F}(\mathbf{r}){:}\ \mathbf{R}^m \Rightarrow \mathbf{R}^\mu,\ \mu < m) \tag{3.1}$$

Then the quantities $\mathbf{r}_c^\circ$ and $\mathbf{r}_\omega^\circ$ are determined from the equations

$$\partial \tilde{W}_c / \partial \mathbf{r} = \mathbf{0} \quad (\tilde{W}_c = W_c + \boldsymbol{\lambda}^T \mathbf{F}) \tag{3.2}$$

$$\partial \tilde{W}_\omega / \partial \mathbf{r} = \mathbf{0} \quad (\tilde{W}_\omega = W_\omega + \boldsymbol{\lambda}^T \mathbf{F}) \tag{3.3}$$

respectively (where λ is a μ-dimensional column vector of undetermined Lagrange multipliers); these equations must be completed by inclusion of relationships (3.1).

Let $\mathbf{r} = \mathbf{r}_c^\circ$, $\boldsymbol{\lambda} = \boldsymbol{\lambda}_c^\circ$ be a solution of system (3.1), (3.2), i.e.

$$\frac{\partial V}{\partial \mathbf{r}} + \frac{\partial C}{\partial \mathbf{r}} + \boldsymbol{\lambda}^T \frac{\partial \mathbf{F}}{\partial \mathbf{r}} = \mathbf{0}, \quad \mathbf{F}(\mathbf{r}) = \mathbf{0} \tag{3.4}$$

for $\mathbf{r} = \mathbf{r}_c^\circ$, $\boldsymbol{\lambda} = \boldsymbol{\lambda}_c^\circ$.

Set

$$\boldsymbol{\omega} = \mathbf{c}\mathbf{D}_0^{-1} \tag{3.5}$$

Here and throughout, the subscript zero means that the quantity in question is evaluated at $\mathbf{r} = \mathbf{r}_c^\circ$.

Consider system (3.1), (3.3)

$$\frac{\partial V}{\partial \mathbf{r}} - \frac{\partial \Omega}{\partial \mathbf{r}} + \boldsymbol{\lambda}^T \frac{\partial \mathbf{F}}{\partial \mathbf{r}} = \mathbf{0}, \quad \mathbf{F}(\mathbf{r}) = \mathbf{0} \tag{3.6}$$

from which the quantities $\mathbf{r}_\omega^\circ$ and $\boldsymbol{\lambda}_\omega^\circ$ are determined. Taking the obvious identity

$$\partial / \partial \mathbf{r}(\mathbf{D}\mathbf{D}^{-1}) \equiv \mathbf{0}$$

into consideration, one can express the second term of the first equation in system (3.6) as

$$-\frac{\partial \Omega}{\partial \mathbf{r}} = \mathbf{D}\frac{\partial \Omega^{-1}}{\partial \mathbf{r}}\mathbf{D}$$

Using formula (3.5), we conclude that this system is satisfied by $\mathbf{r} = \mathbf{r}_c^\circ$, $\boldsymbol{\lambda} = \boldsymbol{\lambda}_c^\circ$. Consequently, $\mathbf{r}_\omega^\circ = \mathbf{r}_c^\circ$, $\boldsymbol{\lambda}_\omega^\circ = \boldsymbol{\lambda}_c^\circ$ (provided that (3.5) holds). We can prove similarly that if $\mathbf{r} = \mathbf{r}_\omega^\circ$, $\boldsymbol{\lambda} = \boldsymbol{\lambda}_\omega^\circ$ is a solution of system (3.6), then $\mathbf{r}_c^\circ = \mathbf{r}_\omega^\circ$, $\boldsymbol{\lambda}_c^\circ = \boldsymbol{\lambda}_\omega^\circ$, where $\mathbf{r}_c^\circ$, $\boldsymbol{\lambda}_c^\circ$ is a solution of system (3.4) and it is assumed that

$$\mathbf{c} = \boldsymbol{\omega}\mathbf{D}_0 \tag{3.7}$$

The analogous correspondence with regard to the stability of SM and REs of systems with symmetry is not complete. More accurately, the following theorem holds.

Theorem 2. If the reduced potential W_ω has a strict local minimum at the point $\mathbf{r} = \mathbf{r}_\omega^\circ$, then the effective potential W_c has a strict local minimum at the point $\mathbf{r} = \mathbf{r}_c^\circ$, provided that $\mathbf{c}$ and $\boldsymbol{\omega}$ satisfy relationship (3.5) ((3.7)).

Proof. Suppose that condition (3.5) ((3.7)) is satisfied. Then $\mathbf{r}_c^\circ = \mathbf{r}_\omega^\circ = \mathbf{r}^\circ$. In that case

$$\begin{aligned}
&(W_c(\mathbf{r}) - W_c(\mathbf{r}^\circ)) - (W_\omega(\mathbf{r}) - W_\omega(\mathbf{r}^\circ)) = C - C_0 + \Omega - \Omega_0 = \\
&= \frac{1}{2}\mathbf{c}(\mathbf{D}^{-1} - \mathbf{D}_0^{-1})\mathbf{c}^T + \frac{1}{2}\boldsymbol{\omega}(\mathbf{D} - \mathbf{D}_0)\boldsymbol{\omega}^T = \\
&= \frac{1}{2}\boldsymbol{\omega}(\mathbf{D}_0\mathbf{D}^{-1}\mathbf{D}_0 - 2\mathbf{D}_0 + \mathbf{D})\boldsymbol{\omega}^T = \frac{1}{2}\mathbf{x}\,\mathbf{D}^{-1}\mathbf{x}^T \geqslant 0 \quad (\mathbf{x} = \boldsymbol{\omega}(\mathbf{D}_0 - \mathbf{D}))
\end{aligned}$$

Consequently, $W_c(\mathbf{r}) - W_c(\mathbf{r}^\circ) \geqslant W_\omega(\mathbf{r}) - W_\omega(\mathbf{r}^\circ)$, i.e. if $W_\omega(\mathbf{r}) > W_\omega(\mathbf{r}^\circ)$, then $W_c(\mathbf{r}) > W_c(\mathbf{r}^\circ)$.

Corollary. If a RE is stable in the secular sense, the corresponding SM is also stable in the secular sense.

Remark. Conditions for the stability of SM cannot be more restrictive than the stability conditions for the corresponding REs, but they may be more inclusive.
Note that the converse is not necessarily true. In particular, a steady motion may be stable in the secular sense even if the corresponding RE is unstable (see the example below).

Let us call a steady motion (RE) *trivial* if $\mathbf{r}^\circ_c$ is independent of $\mathbf{c}$ ($\mathbf{r}^\circ_\omega$ is independent of $\boldsymbol{\omega}$).
Obviously (see (3.4) and (3.6), respectively), trivial SM (TSM) and trivial REs (TREs) satisfy the relations

$$\frac{\partial V}{\partial \mathbf{r}} + \boldsymbol{\lambda}^T \frac{\partial \mathbf{F}}{\partial \mathbf{r}} = \mathbf{0}, \quad \mathbf{F}(\mathbf{r}) = \mathbf{0} \tag{3.8}$$

that is, they are always identical. When that happens, we have

$$\partial C / \partial \mathbf{r} = \mathbf{0}, \quad \partial \Omega / \partial \mathbf{r} = \mathbf{0} \tag{3.9}$$

on TSM and TREs, identically with respect to $\mathbf{c}$ and $\boldsymbol{\omega}$, respectively.

Theorem 3. The indices of the quadratic forms (1.7) and (2.6) for TSM and the corresponding TREs are identical.
The proof follows in an obvious manner from (3.5)–(3.9).

Corollary 1. The degree of instability of a TSM is always the same as that of the corresponding TRE.
Let us call a steady motion (RE) non-degenerate if the determinant of the matrix of the second variation of W_c (W_ω) on the steady motion (on the RE) does not vanish; otherwise we shall say that the steady motion (RE) is degenerate.

Corollary 2. The conditions for secular stability of non-degenerate TSM and the corresponding TREs are always the same.

Remarks. 1. The conditions for stability of degenerate TSM cannot be more restrictive than those for the stability of the corresponding degenerate TREs, but they may be more inclusive (see example).
2. The degree of instability of non-degenerate non-trivial SM cannot exceed that of the corresponding non-trivial REs, but they may be smaller (see example).
3. If $\mathbf{r}$ and $\mathbf{s}$ are true positional and cyclic coordinates and $\mathbf{v} = (\dot{\mathbf{r}}, \dot{\mathbf{s}})$, the results presented here agree with previously published results [13–17].

4. The results presented above can be extended in a natural way to the case of dissipative systems that admit of first integrals (1.2) and satisfy the energy relation $dH/dt \leqslant 0$ (instead of the energy integral (1.1)), and also to the case in which the equations $\delta W_c = 0$ ($\delta W_\omega = 0$) define not a point $\mathbf{r} = \mathbf{r}^\circ(\mathbf{c})$ ($\mathbf{r} = \mathbf{r}^\circ(\boldsymbol{\omega})$) but a compact set $\mathbf{M}_0(\mathbf{c}) \subset \mathbf{M}$ ($\mathbf{M}_0(\boldsymbol{\omega}) \subset \mathbf{M}$).
Indeed, for dissipative systems, proceeding as before, one can introduce an effective potential $W_c(\mathbf{r})$ and reduced potential $W_\omega(\mathbf{r})$ by formulae (1.7) and (2.5), respectively. When that is done, it follows from earlier results [18, 19] that the sets on which $\delta W_c = 0$ or $\delta W_\omega = 0$ define invariant sets (ISs) of the free or restricted dissipative system, respectively. If it is also true that the set $\mathbf{M}_0(\mathbf{c})$ ($\mathbf{M}_0(\boldsymbol{\omega})$) is compact and provides function $W_c(\mathbf{r})$ ($W_\omega(\mathbf{r})$) with a strict local minimum, then $\mathbf{M}_0(\mathbf{c})$ (($\mathbf{M}_0(\boldsymbol{\omega})$) is a stable IS [18, 19].
Analogous reasoning implies the following theorems.

Theorem 4. For any set $\mathbf{c}$ of constants of Noether integrals (for any set of constants $\boldsymbol{\omega}$ defining the velocity of motion of the system along its symmetry group), a set of constants $\boldsymbol{\omega}$ (a set of constants $\mathbf{c}$) exists such that the set $\mathbf{M}_0(\mathbf{c})$ of solutions of system (1.5) is identical with the set $\mathbf{M}_0(\boldsymbol{\omega})$ of solutions of system (2.3).

Theorem 5. If the reduced potential W_ω has a strict local minimum on the set $\mathbf{M}_0(\boldsymbol{\omega})$, then the effective potential $\mathbf{M}_c$ has a strict local minimum on the set $\mathbf{M}_0(\mathbf{c})$, provided that $\mathbf{c}$ and $\boldsymbol{\omega}$ satisfy relationship (3.5) (or (3.7)).

Corollary. If the set $\mathbf{M}_0(\boldsymbol{\omega})$ is stable in the secular sense, then the set $\mathbf{M}_0(\mathbf{c})$ is also stable in the secular sense.

Note that $W_c(\mathbf{r})$ ($W_\omega(\mathbf{r})$) will certainly have a strict local minimum on the set $\mathbf{M}_0(\mathbf{c})$ ($\mathbf{M}_0(\boldsymbol{\omega})$), provided that the second variation $\delta^2 W_c(\mathbf{M}_0(\mathbf{c}))$ ($\delta^2 W_\omega(\mathbf{M}_0(\boldsymbol{\omega}))$ is a positive-definite function of the deviations of the vector $\mathbf{r}$ from the IS $\mathbf{M}_0(\mathbf{c})$ ($\mathbf{M}_0(\boldsymbol{\omega})$).

As before, we shall call an IS $\mathbf{M}_0(\mathbf{c})$ ($\mathbf{M}_0(\boldsymbol{\omega})$) trivial (TIS) if it does not change when the parameter $\mathbf{c}$ (the parameter $\boldsymbol{\omega}$) is varied. Obviously, a TIS $\mathbf{M}_0(\mathbf{c})$ corresponds to the TIS $\mathbf{M}_0(\boldsymbol{\omega})$ and conversely. We thus have the following theorem.

Theorem 6. The indices of the quadratic forms (1.7) and (2.6) for a TIS $\mathbf{M}_0(\mathbf{c})$ and the corresponding TIS $\mathbf{M}_0(\boldsymbol{\omega})$ are identical.

Corollaries. 1. A TIS $\mathbf{M}_0(\mathbf{c})$ and the corresponding TIS $\mathbf{M}_0(\boldsymbol{\omega})$ have the same degrees of instability.

2. The conditions for secular stability of a non-degenerate TIS $\mathbf{M}_0(\mathbf{c})$ and for that of the corresponding TIS $\mathbf{M}_0(\boldsymbol{\omega})$ are always the same.

Remark. If a compact invariant set $\mathbf{M}_0(\mathbf{c})$ ($\mathbf{M}_0(\boldsymbol{\omega})$) is isolated for fixed values of the parameters $\mathbf{c}$ ($\boldsymbol{\omega}$) from the set $H \equiv 0$, then, by earlier results [18, 19], this set is partially asymptotically stable (unstable) provided that it provides (does not provide) the function $W_c(\mathbf{r})$ ($W\omega(\mathbf{r})$) with a strict local minimum (even a non-strict minimum). In that case the TIS $\mathbf{M}_0(\mathbf{c})$ and $\mathbf{M}_0(\boldsymbol{\omega})$ are either both partially asymptotically stable or both unstable.

5. We will now illustrate the results, applying them to the motion of a heavy dynamically symmetric sphere along a horizontal plane with sliding friction.

Let m be the sphere mass, $\boldsymbol{\theta} = \mathrm{diag}\,(A, A, C)$ be the central inertia tensor, a be the displacement of the geometrical centre O of the sphere from its mass centre G, r be the radius of the sphere, γ be the unit vector of the upward vertical, $\mathbf{e}$ be the unit vector of the dynamic axis of symmetry of the sphere, which points along the vector GO, $\mathbf{v}$ be the velocity of the sphere's mass centre, $\boldsymbol{\omega}$ be the angular velocity of rotation of the sphere about its mass centre, and g be the acceleration due to gravity.

If the only forces acting on the sphere are gravity and sliding friction, the system admits of an energy relation

$$\frac{dH}{dt} \leq 0, \quad H = \frac{1}{2}(m\mathbf{v}, \mathbf{v}) + \frac{1}{2}(\boldsymbol{\theta}\boldsymbol{\omega}, \boldsymbol{\omega}) - mg(\boldsymbol{\rho}, \boldsymbol{\gamma})$$

and a Jellett integral

$$J = (\boldsymbol{\theta}\boldsymbol{\omega}, \boldsymbol{\rho}) = rj$$

where $\boldsymbol{\rho} = -r\boldsymbol{\gamma} + a\mathbf{e} = GK$, K is the point at which the sphere touches the support plane and j is an arbitrary constant.

The effective potential of the system is defined by the following relation [20]

$$W_j = \min_{\mathbf{v}, \boldsymbol{\omega}} H|_{J=rj} = -mga\cos\vartheta + \frac{1}{2}\frac{j^2}{A\sin^2\vartheta + C(\cos\vartheta - \varepsilon)^2} + \text{const}$$

$$\varepsilon = a/r, \quad \cos\vartheta = (\boldsymbol{\gamma}, \mathbf{e})$$

A minimum is reached at $\mathbf{v} = \mathbf{0}$, $\boldsymbol{\omega} = \omega\boldsymbol{\rho}/r$, where

$$\omega = \omega(j) = \frac{r^2}{(\boldsymbol{\theta}\boldsymbol{\rho}, \boldsymbol{\rho})} j \tag{5.1}$$

If it is assumed that $(\boldsymbol{\omega}, \boldsymbol{\rho})r/(\boldsymbol{\rho}, \boldsymbol{\rho}) = \omega$ on all motions of the sphere, where ω is an arbitrary constant, then the reduced potential is defined by

$$W_\omega = V - T|_{\mathbf{v}=0, \boldsymbol{\omega}=\omega\boldsymbol{\rho}/r} = -mga\cos\vartheta - \tfrac{1}{2}(A\sin^2\vartheta + C(\cos\vartheta - \varepsilon)^2)\omega^2$$

By the results presented previously, the set of critical points $\vartheta = \vartheta^\circ(j)$ of W_j coincides with the set of critical points $\vartheta = \vartheta^\circ(\omega)$ of W_ω if the constants j and ω are related as (see (5.1))

$$j = (A\sin^2\vartheta^\circ + C(\cos\vartheta^\circ - \varepsilon)^2)\omega \tag{5.2}$$

Obviously, the set of critical points of W_ω contains two trivial branches $\vartheta_1 = 0$, $\vartheta_2 = \pi$ and one non-trivial branch $\vartheta_3 = \vartheta_3(\omega^2)$. The latter is defined by

$$\omega^2 = \frac{mga}{C(\varepsilon-(1-\delta)\cos\vartheta)} \quad \left(\delta = \frac{A}{C}\right) \tag{5.3}$$

The critical points $\vartheta_1 = 0$ and $\vartheta_2 = \pi$ correspond to vertical rotations of the sphere and the set of critical points $\vartheta = \vartheta_3$ corresponds to regular precessions of the sphere. The function W_ω then has a minimum (maximum) at the points ϑ_1, ϑ_2 and ϑ_3 under the conditions

$$C\omega^2(1-\delta-\varepsilon) + mga > 0 \quad (<0) \tag{5.4}$$

$$C\omega^2(1-\delta+\varepsilon) - mga > 0 \quad (<0) \tag{5.5}$$

$$\delta - 1 > 0 \quad (<0) \tag{5.6}$$

The set of critical points of W_j also contains two trivial branches $\vartheta_1 \equiv 0$ and $\vartheta_2 \equiv \pi$ and at most two non-trivial branches $\vartheta_{3,4} = \vartheta_{3,4}(j^2)$. The latter are defined by the relation

$$j^2 = Cmga\frac{(\delta\sin^2\vartheta + (\cos\vartheta - \varepsilon)^2)^2}{(\delta-1)\cos\vartheta + \varepsilon} \tag{5.7}$$

Note that Eq. (5.3) is uniquely solvable for $\vartheta(\omega^2)$, while Eq. (5.7) may have at most two groups of solutions $\vartheta(j^2)$; in the configuration space S^2, however, these relations (taking (5.2) into account) define the same set of critical points of the functions W_ω and W_j, respectively.

By our previous results, the conditions

$$\frac{j^2}{C(1-\varepsilon)^4}(1-\delta-\varepsilon) + mga > 0 \quad (<0) \tag{5.8}$$

$$\frac{j^2}{C(1+\varepsilon)^4}(1-\delta+\varepsilon) - mga > 0 \quad (<0) \tag{5.9}$$

obtained from (5.4) and (5.5) provided that $\vartheta^\circ = 0$ and $\vartheta^\circ = \pi$, respectively (see (5.2)), define conditions for a minimum (maximum) of W_j at the points $\vartheta_1 = 0$ and $\vartheta_2 = \pi$, respectively. Analogous conditions were obtained previously (see, for example, [20]) by direct investigation of the function W_j.

As to conditions (5.6), it follows from our results that for $\delta > 1$ the function W_j will certainly have a minimum on the unique (for $\delta > 1$) non-trivial branch $\vartheta_3 = \vartheta_3(j^2)$. However, this function may also have a minimum when $\delta < 1$ (see, for example, [20]), unlike the function W_ω.

Thus, the stability of vertical rotations of a free heavy dynamically symmetric sphere on a plane with sliding friction (the stability of a TSM) is defined both by (5.4) or (5.5) (and then ω is the angular velocity of rotation) and by (5.8) or (5.9) (and then j is the constant of Jellett integral, normalized to the sphere radius). Regular precessions of the sphere (non-trivial SMs) are always stable if the inertia ellipsoid is prolate along the axis of symmetry (see (5.6)), but it may be stable even if the ellipsoid is oblate. At the same time, regular precessions of the sphere, whose motion satisfies the relations $(\boldsymbol{\omega}, \boldsymbol{\rho})r/(\boldsymbol{\rho}, \boldsymbol{\rho}) = \omega$, are stable (unstable) if the inertia ellipsoid of the sphere is prolate (oblate).

It is noteworthy that in investigating the motion of a top on a plane with friction one cannot directly use existing results [13–17], because, first, the system is dissipative, and, second, apart from zero-dimensional invariant sets (rotations about the axis of symmetry), the system also admits of one-dimensional invariant sets (regular precessions). The fact that the system is dissipative does not play an essential part and is readily taken into consideration, whereas if one abandons the invariant description of the motion of the top and instead transforms to some kind of generalized coordinates, the motions of the system cannot all be described in a uniform setting, and it becomes much more difficult to investigate the problem. Indeed, one cannot investigate vertical rotations in terms of Euler angles, while Krylov angles oblige one to disregard regular precessions.

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